# Greedy `MaxCut` Algorithms and their Information Content

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# Contents

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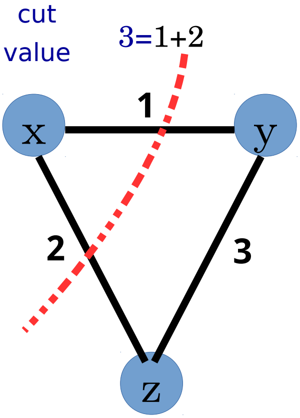

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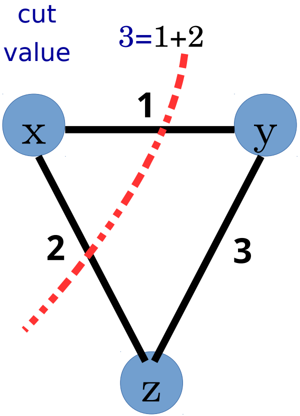

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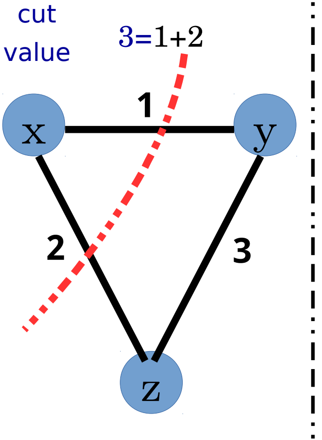

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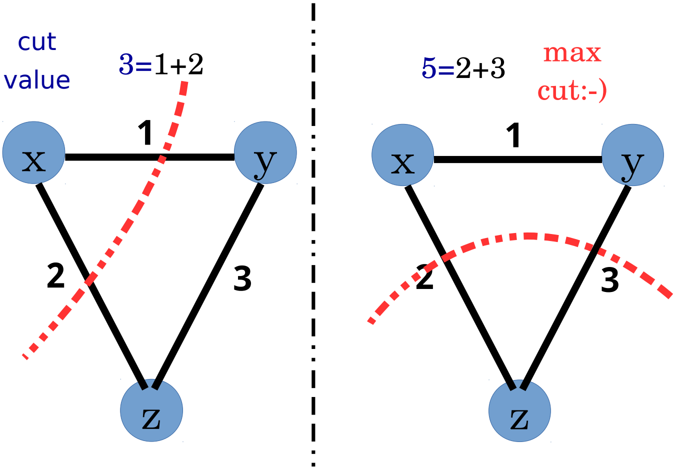

# Greedy Algorithms for MaxCut

| Name | Greedy Heuristic | Techniques | |
|---|---|---|---|
| | | Sorting | Init. Vertices |
| Deterministic Double Greedy | Double | | |
| SG (Sahni & Gonzales) | | | ✓ |
| SG3 (variant of SG) | | ✓ | ✓ |
| Edge Contraction (EC) | Backward | ✓ | |

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**for** each vertex $v_i \in V$ **do**
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- for each vertex, decides whether it should be added to $S$, or removed from $T$

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    if a_i ≥ b_i then
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Differences between the double greedy algorithms:

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| D2Greedy | $\rightarrow$ | select the first 2 vertices | $\rightarrow$ | SG |
| SG | $\rightarrow$ | sort the candidates | $\rightarrow$ | SG3 |

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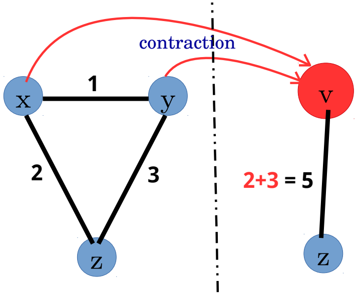

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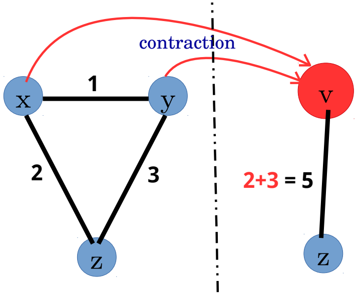

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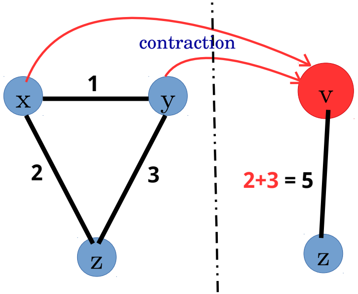

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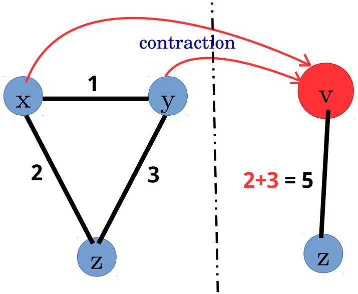

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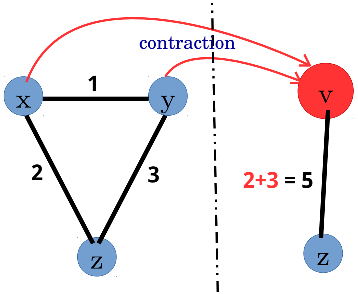

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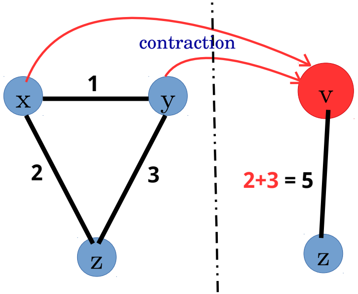


Backward greedy: EC tries to remove the lightest edge from the cut set in each step

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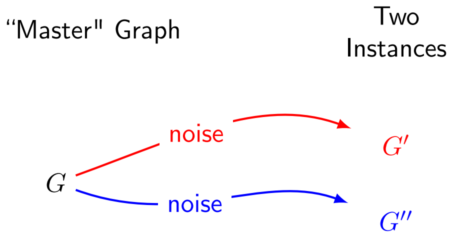

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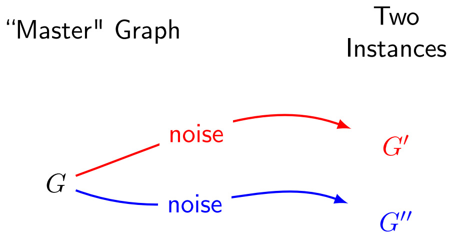


- Models/algorithms should generalize well from $G'$ to $G''$

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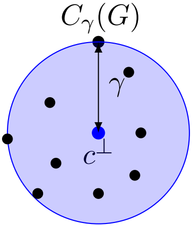

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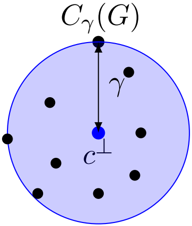


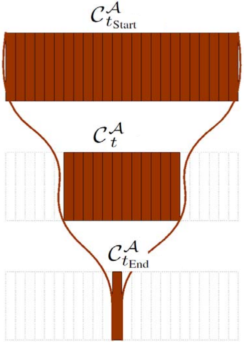

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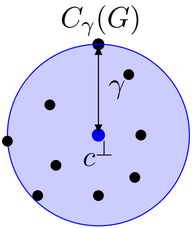


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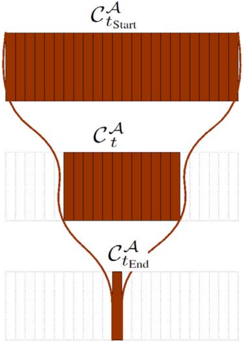


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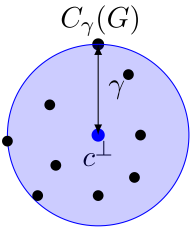


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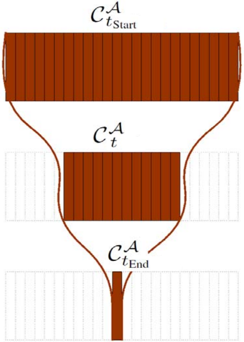


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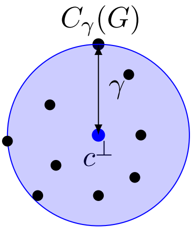


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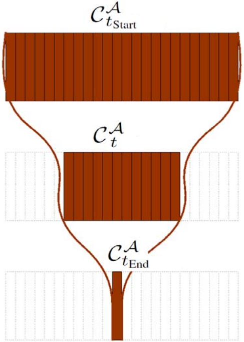


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## Analogical mutual information in step $t$

$$I_t^{\mathscr{A}} := \mathbb{E}_{G',G''}\left[\log\left(\frac{|\mathcal{C}|\cdot|\Delta C_t^{\mathscr{A}}(G',G'')|}{|C_t^{\mathscr{A}}(G')|\cdot|C_t^{\mathscr{A}}(G'')|}\right)\right]$$

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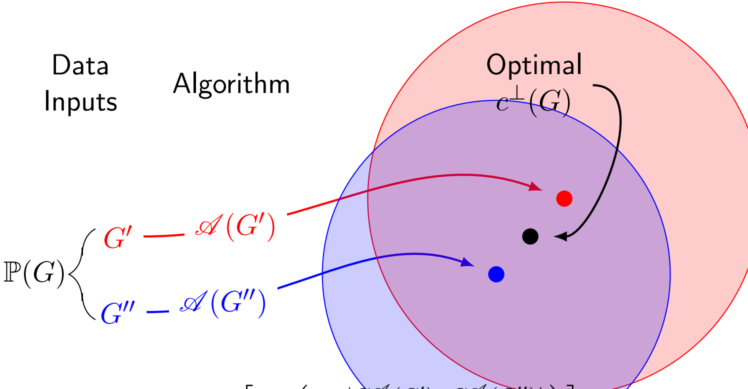


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(stepwise information)

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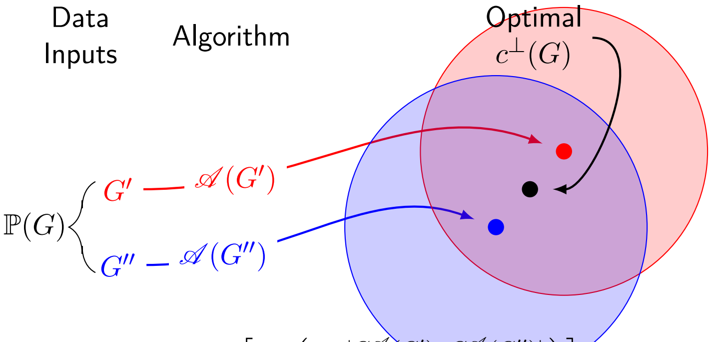


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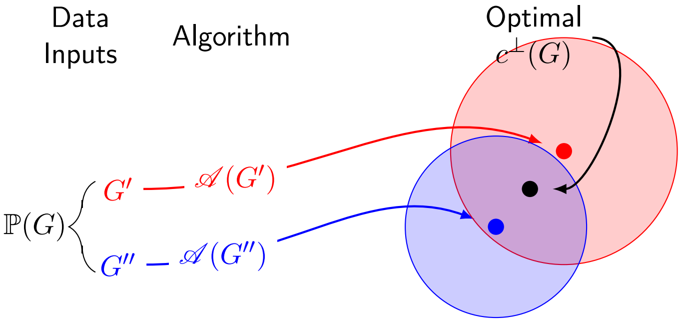


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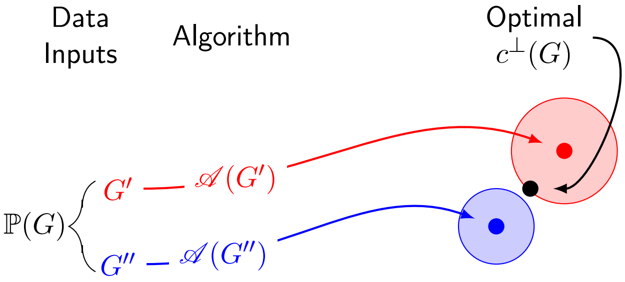


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(stepwise information)

$\nearrow$ step $t \Leftrightarrow \searrow$ resolution $\gamma$

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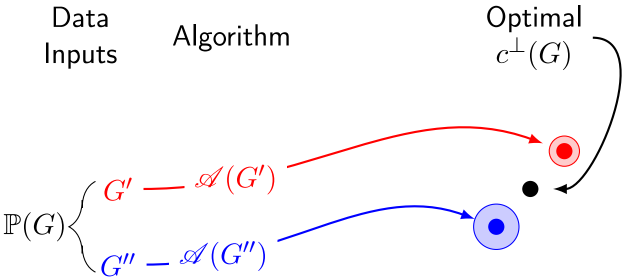


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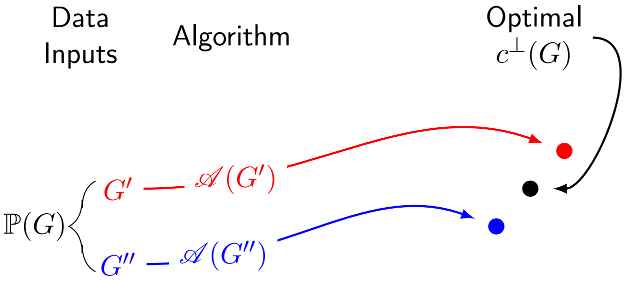


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  We propose (and prove correctness) polynomial time algorithm to count (not going in detail here):

For the SG3 (Alg. 6, see Supplement), after step $t$ ($t = 1, \cdots, n-1$) there are $k = n - t - 1$ unlabelled vertices, and it is clear that $|C(G')| = |C(G'')| = 2^k$.

To count the intersection set $\Delta(G', G'')$, assume the solution set pair of $G'$ is $(S_1', S_2')$, the solution set pair of $G''$ is $(S_1'', S_2'')$, so the unlabelled vertex sets are $T' = V \backslash \{S_1' \cup S_2'\}$, $T'' = V \backslash \{S_1'' \cup S_2''\}$, respectively. Denote $L := T' \cap T''$ be the common vertices of the two unlabelled vertex sets, so $l = |L|$ ($0 \le l \le k$) is the number of common vertices in the unlabelled $k$ vertices. Denote $M' := T' \backslash L$, $M'' := T'' \backslash L$ be the sets of different vertex sets between the two unlabelled vertex sets. Then,

$$\Delta(G', G'') = \begin{cases} 2^l & \text{if } (S_1'' \backslash M', S_2'' \backslash M') \text{ is matched by} \\ & (S_1' \backslash M'', S_2' \backslash M'') \text{ or } (S_2' \backslash M'', S_1' \backslash M'') \\ 0 & \text{otherwise} \end{cases}$$

# Counting – Edge Contraction Algorithm

- In step $t$, there are $k$ "**super**" vertices, get $|C_t^{\mathscr{A}}(G^{'})| = |C_t^{\mathscr{A}}(G^{''})| = 2^{k-1} - 1$

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---

**Algorithm 3:** Common Super Vertex Counting

**Input**: Two distinct super vertex sets $P$, $Q$
**Output**: Maximum number of common super vertices after all possible contractions

$c := 0$;
**while** $P \neq \emptyset$ **do**
  Randomly pick $\mathbf{p}_i \in P$;
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**Theorem 1.** *Given two distinct super vertex sets* $P := \{\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_h\}$, $Q := \{\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_h\}$ *(any 2 super vertices inside* $P$ *or* $Q$ *do not intersect, and there is no common super vertex between* $P$ *and* $Q$*), such that* $\mathbf{p}_1 \cup \mathbf{p}_2 \cup \cdots \cup \mathbf{p}_h = \mathbf{q}_1 \cup \mathbf{q}_2 \cup \cdots \cup \mathbf{q}_h$*, Alg. 3 returns the maximum number of common super vertices between* $P$ *and* $Q$ *after all possible contractions.*

# Contents

# Noise Model: Gaussian Edge Weights

**Master Graph** $G$
Gaussian distributed edge weights:

$$W_{ij} \sim N(\mu, \sigma_m^2), \mu = 600, \sigma_m = 50$$

Negative edges are set to be $\mu$.

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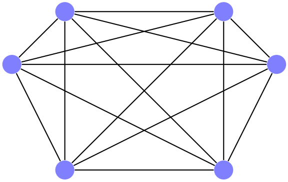

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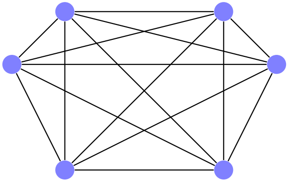

Master graph $G$ with Gaussian weights

**Noisy Graphs $G^{'}$, $G^{''}$**
$G^{'}$, $G^{''}$ are obtained by adding Gaussian distributed noise.
Negative edges are set to be 0.

# Noise Model: Edge Reversal

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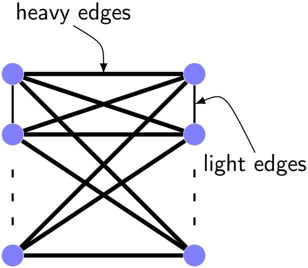


Approximate bipartite graph $G'_b$

**Master Graph** $G$

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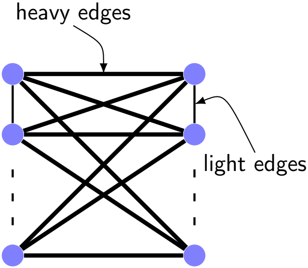


Approximate bipartite graph $G'_b$

# Noise Model: Edge Reversal

**Master Graph $G$**

1. approximate bipartite $G_b'$: *light edges*, *heavy edges*
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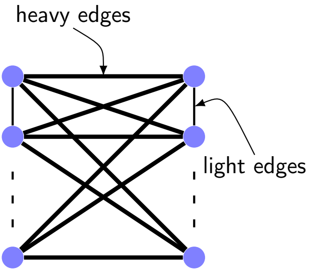


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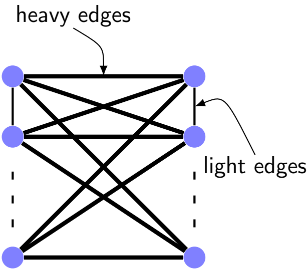



`Approximate bipartite graph` $G'_b$

**Noisy Graphs** $G^{'}$, $G^{''}$

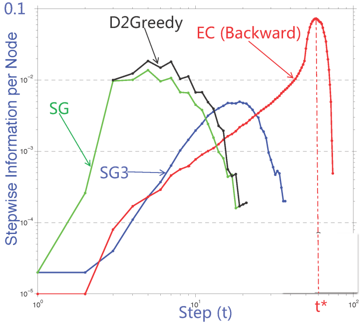
- Flip $G \Rightarrow G^{'}$ and $G^{''}$.
  Probability of flipping an edge: Bernoulli distribution with $p$,

$$(\text{flip } e_{ij}) \sim \text{Ber}(p)$$

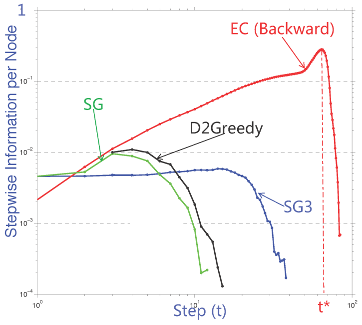
$p$: noise level

# Stepwise Information $I_t^{\mathscr{A}}$

$$I_t^{\mathscr{A}} := \mathbb{E}_{G',G''}\left[\log\left(\frac{|\mathcal{C}|\cdot|\Delta C_t^{\mathscr{A}}(G',G'')|}{|C_t^{\mathscr{A}}(G')|\cdot|C_t^{\mathscr{A}}(G'')|}\right)\right]$$




Gaussian Model, $\sigma = 125$

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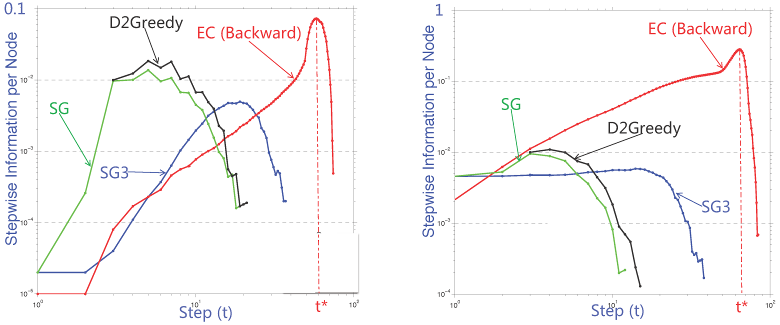


Gaussian Model, $\sigma = 125$ Edge Reversal, $p = 0.65$

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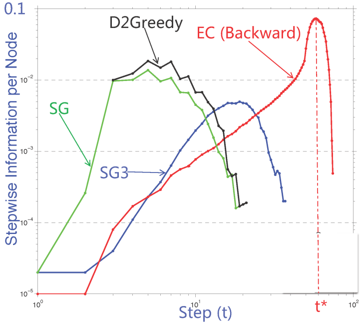


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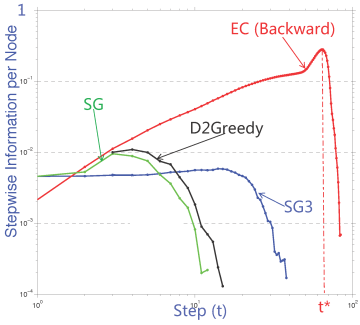
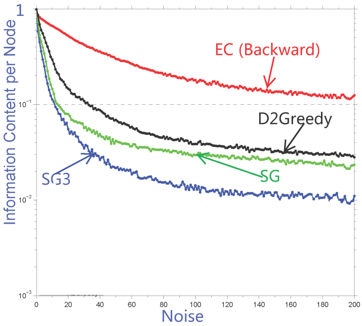



Edge Reversal, $p = 0.65$

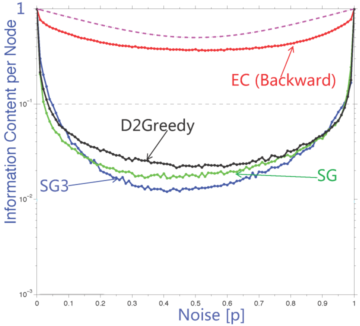
- $I_t^{\mathscr{A}}$ behavior: increase initially $\Rightarrow$ reach the optimal step $t^*$ $\Rightarrow$ decreases $\Rightarrow$ vanishes.
- consistent with analysis: $\nearrow t \Rightarrow$ tradeoff of roubstness and informativeness

# Information Content $I^{\mathscr{A}}$

$I^{\mathscr{A}} := \max_t I_t^{\mathscr{A}}$ (channel capacity)




Gaussian Edge Weights Model

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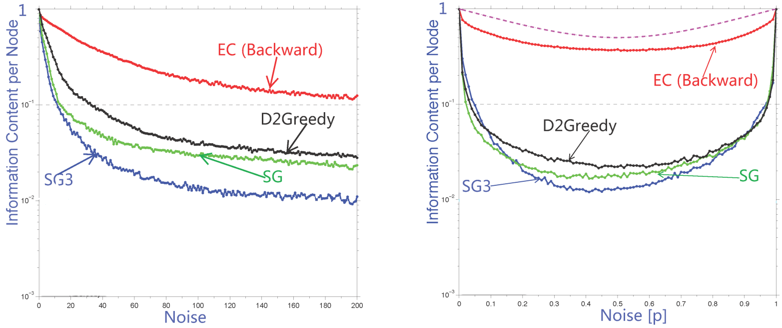


Gaussian Edge Weights Model        Edge Reversal Model

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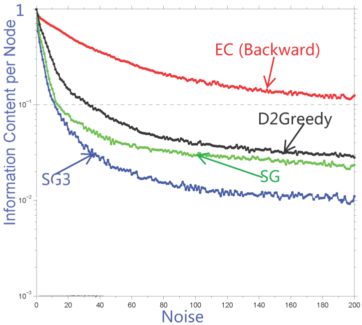


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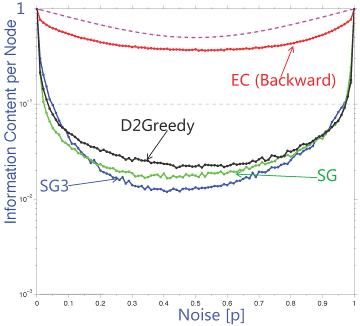
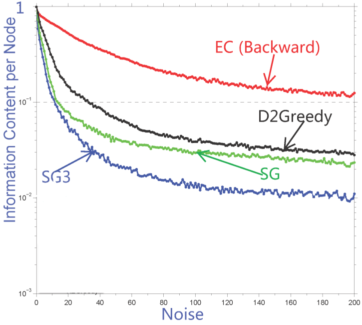



Edge Reversal Model

- All reach max. information content in the noise free limit $(G' = G'')$ $(p = 0, 1$ in edge reversal model, $\sigma = 0$ in Gaussian model)
- 1 node transmits about 1 bit information

# Effect of Greedy Heuristics

Backward greedy $\succcurlyeq$ double greedy

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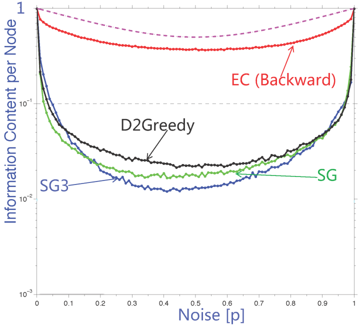


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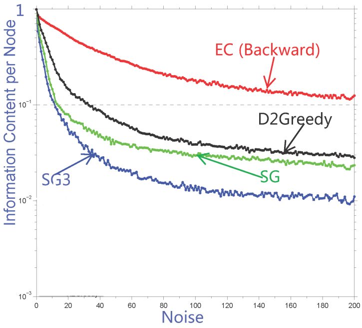


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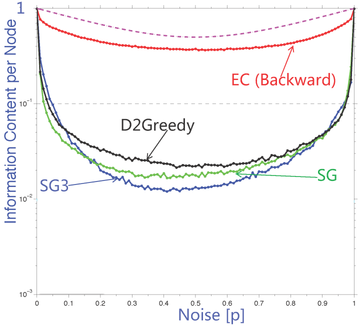


**Edge Reversal Model**

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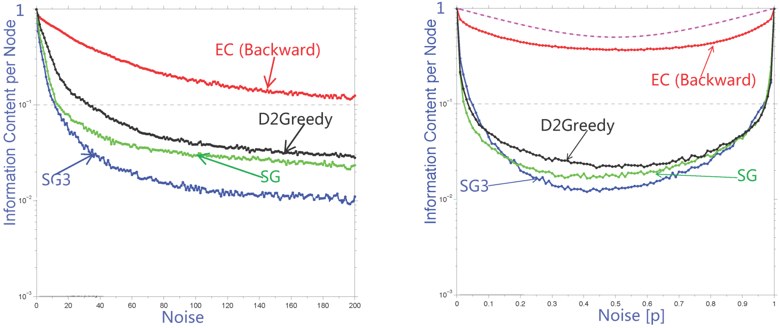
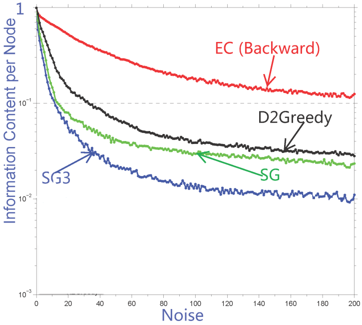



Gaussian Edge Weights Model    Edge Reversal Model

- Delayed decision making of backward greedy
- EC preserves consistent solutions by contracting lightest edge (having low probability to be included in the cut)

# Effect of Greedy Techniques

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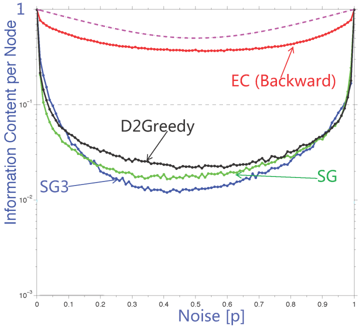


Edge Reversal Model

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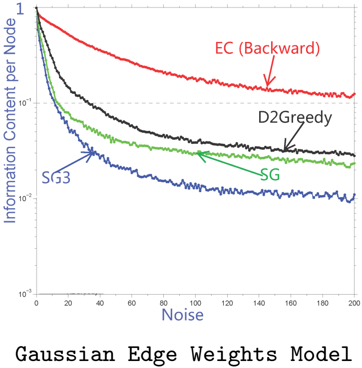


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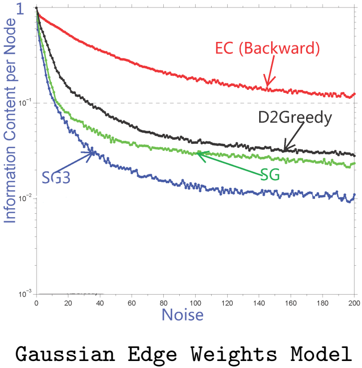


- Initializing (D2Greedy ⇒ SG): ↘, due to early decision making
- Sorting candidates (SG ⇒ SG3): ↘, due to early decision making

# Discussion

- **Observation**:
  Different greedy heuristics (backward, double) and different processing techniques (sorting candidates, initializing the first 2 vertices) sensitively influence the information content of $\mathscr{A}$.

# Discussion

- **Observation**:
  Different greedy heuristics (backward, double) and different processing techniques (sorting candidates, initializing the first 2 vertices) sensitively influence the information content of $\mathscr{A}$.
- **Conjecture**:
  Backward greedy $\overset{\text{delayed decision making}}{\succeq}$ double greedy
  for different noise models and noise levels.

# Thank you!

Qs?

# Supplement: Analogy of Communication System

Imaginary communication system:

- message: permutations $\sigma_s \in \Sigma$ on the data space
- encoder: encoding $\sigma_s$ using $C_t^{\mathscr{A}}(\sigma_s \circ G')$ (codebook vector)
- channel: noisy instances $G'$, $G''$
- decoder: max. overlap of approx. sets:
  $\hat{\sigma} := \arg\max_{\sigma \in \Sigma} |C_t^{\mathscr{A}}(\sigma \circ G'') \cap C_t^{\mathscr{A}}(\sigma_s \circ G')|$

## Analogical mutual information in step $t$

$$I_t^{\mathscr{A}}(\sigma_s; \hat{\sigma}) := \mathbb{E}_{G', G''}\left[\log\left(|\mathcal{C}| \frac{|C_t^{\mathscr{A}}(G') \cap C_t^{\mathscr{A}}(G'')|}{|C_t^{\mathscr{A}}(G')| \cdot |C_t^{\mathscr{A}}(G'')|}\right)\right]$$

channel capacity $I^{\mathscr{A}} := \max_t I_t^{\mathscr{A}}$ (Information content of $\mathscr{A}$ )